

Closed form solution of non-homogeneous equations with Toeplitz plus Hankel operators

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Abstract

Considered is the equation

$$(T(a) + H(b))\phi = f, \quad (\star)$$

where $T(a)$ and $H(b)$, $a, b \in L^\infty(\mathbb{T})$ are, respectively, Toeplitz and Hankel operators acting on the classical Hardy spaces $H^p(\mathbb{T})$, $1 < p < \infty$. If the generating functions a and b satisfy the so-called matching condition [1,2],

$$a(t)a(1/t) = b(t)b(1/t), \quad t \in \mathbb{T},$$

an efficient method for solving equation (\star) is proposed. The method is based on the Wiener–Hopf factorization of the scalar functions $c(t) = a(t)b^{-1}(t)$ and $d(t) = a(t)b^{-1}(1/t)$ and allows one to find all solutions of the equations mentioned.

1 Introduction

Let $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$ be the counterclockwise oriented unit circle in the complex plane \mathbb{C} , and let $L^p = L^p(\mathbb{T})$, $1 \leq p \leq \infty$ denote the space of all Lebesgue measurable functions f such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(t)|^p |dt| \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := \operatorname{ess\,sup}_{t \in \mathbb{T}} |f(t)| < \infty.$$

If $f \in L^1$ then, as usual, we use the notation \widehat{f}_n to denote the n -th Fourier coefficient of f , i.e.

$$\widehat{f}_n := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta, \quad n \in \mathbb{Z}.$$

The Hardy spaces $H^p := H^p(\mathbb{T})$ and $\overline{H^p} := \overline{H^p(\mathbb{T})}$, $1 \leq p \leq \infty$ are defined as follows

$$H^p := \{f \in L^p : \widehat{f}_n = 0 \text{ for all } n < 0\},$$

$$\overline{H^p} := \{f \in L^p : \widehat{f}_n = 0 \text{ for all } n > 0\}.$$

Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ denote the set of all non-negative integers. Consider the operator P defined by

$$P : \sum_{n \in \mathbb{Z}} \widehat{f}_n t^n \rightarrow \sum_{n \in \mathbb{Z}_+} \widehat{f}_n t^n,$$

and let $Q := I - P$, where I is the identity operator. It is easily seen that P and Q are complimentary projections and it is well-known that they are bounded on any space L^p , $p \in (1, \infty)$. Along with the operators P and Q we also consider the operator $J : L^p \rightarrow L^p$,

$$(Jf)(t) := t^{-1} f(t^{-1}), \quad t \in \mathbb{T}.$$

For $a, b \in L^\infty$, the Toeplitz plus Hankel operator $T(a) + H(b) : H^p \rightarrow H^p$ is defined by

$$T(a) + H(b) := PaP + PbQJ. \tag{1}$$

Various properties of these operators have been established in literature. Special interest represents the Fredholmness of the operator $T(a) + H(b)$ and investigation of its kernel and cokernel. In the case of piecewise continuous generating functions a and b , Fredholm properties of the operator (1) can be derived by a direct application of results [2, Sections 4.95-4.102], [11, Sections 4.5 and 5.7], [12]. The case of quasi piecewise continuous generating functions has been studied in [14], whereas formulas for the index of the operators (1), considered on different Banach and Hilbert spaces

and with various assumptions about the generating functions a and b , have been established in [3, 13]. Recently, progress has been made in computation of defect numbers $\dim \ker(T(a) + H(b))$ and $\dim \operatorname{coker}(T(a) + H(b))$ for various classes of generating functions a and b [1, 5]. The more delicate problem of the description of the spaces $\ker(T(a) + H(b))$ and $\operatorname{coker}(T(a) + H(b))$ has been considered [5, 6]. In particular, for generating functions a and b , satisfying the so-called matching condition (see condition (12) below), explicit and efficient formulas for the elements of the kernel of the operator $T(a) + H(b)$ have been established. Thereby, for this class of generating functions, homogeneous equations with Toeplitz plus Hankel operators can be effectively solved. On the other hand, non-homogeneous equations with Toeplitz plus Hankel operators and also equations with Wiener–Hopf plus Hankel operators often arise in applications [7, 8, 10] but, as a rule, only some approximation methods of their solutions have been systematically studied so far.

The aim of this work is to present a method for solution of the operator equations

$$(T(a) + H(b))\varphi = f, \quad f \in L^p, \quad 1 < p < \infty, \quad (2)$$

with generating functions $a, b \in L^\infty$ satisfying relation (12). Note that this relation has been first used in [1] when studying the dimension of kernels and cokernels of Toeplitz plus Hankel operators with piecewise continuous generating functions a and b . Regardless of [1], the importance of relation (12) for the investigation of Toeplitz plus Hankel operators has been mentioned in [3, Remark 9].

The approach proposed is also applicable to non-homogeneous equations with Wiener–Hopf plus Hankel operators with generating matching functions. However, at the moment there is no description for the kernels of such operators. Nevertheless, an investigation of Wiener–Hopf plus Hankel operators in the situation mentioned, has been started in [4]. As soon as that work will be completed, relevant results can be also used for solution of non-homogeneous Wiener–Hopf plus Hankel equations. The details of the corresponding study will be presented elsewhere.

2 Toeplitz plus Hankel equations and equations with matrix Toeplitz operators

In this section we establish connections between the solutions of Toeplitz plus Hankel equation (2) and solutions of an equations with a matrix Toeplitz operator. On the space $H^p \times H^p$, $1 < p < \infty$, let us consider the operator \mathcal{R} ,

$$\mathcal{R} := \operatorname{diag}(T(a) + H(b), T(a) - H(b)),$$

and let \mathcal{P} and \mathcal{Q} denote the operators

$$\mathcal{P} := \operatorname{diag}(P, P), \quad \mathcal{Q} := \operatorname{diag}(Q, Q),$$

acting on the space $L^p(\mathbb{T}) \times L^p(\mathbb{T})$, $1 < p < \infty$. For any element $a \in L^\infty$, set $\tilde{a}(t) := a(1/t)$. Let $C = C(a, b)$ and $V = V(a, b)$ be 2×2 -matrices,

$$C(a, b) := \begin{pmatrix} 1 & 0 \\ \tilde{b} & \tilde{a} \end{pmatrix}, \quad V(a, b) := \begin{pmatrix} a - b\tilde{b}\tilde{a}^{-1} & b\tilde{a}^{-1} \\ -\tilde{b}\tilde{a}^{-1} & \tilde{a}^{-1} \end{pmatrix},$$

and let $\mathcal{J}, A_1, A_2, B, R$ be the operators defined by

$$\mathcal{J} := \frac{1}{2} \begin{pmatrix} I & J \\ I & -J \end{pmatrix}, \quad A_1 := \text{diag}(I, I) - \text{diag}(P, Q) \begin{pmatrix} a & b \\ \tilde{b} & \tilde{a} \end{pmatrix} \text{diag}(Q, P),$$

$$A_2 := \text{diag}(I, I) + \mathcal{P}V(a, b)\mathcal{Q},$$

$$B := \mathcal{P}V(a, b)\mathcal{P} + \mathcal{Q},$$

$$R := \mathcal{R} + \mathcal{Q}.$$

According to the relation (3.4) of [6], the operators R and B are connected as follows

$$\mathcal{J}^{-1}R\mathcal{J} = A_1A_2BC. \quad (3)$$

Lemma 2.1 *Let $y = (g, h)^T \in H^p(\mathbb{T}) \times H^p(\mathbb{T})$. If $x_0 = (\varphi, \psi)^T \in H^p(\mathbb{T}) \times H^p(\mathbb{T})$ is a solution of the equation*

$$\mathcal{R}x = y, \quad (4)$$

then the element

$$X_0 = \left(\varphi + \psi, P(\tilde{b}(\varphi + \psi) + \tilde{a}J(\varphi - \psi)) \right)^T \quad (5)$$

is a solution of the equation

$$\mathcal{P}V(a, b)\mathcal{P}X = Y, \quad (6)$$

where $Y = \mathcal{P}A_2^{-1}A_1^{-1}\mathcal{J}^{-1}y$.

Proof. If $x_0 = (\varphi, \psi)^T$ is a solution of (4), then one also has

$$(\mathcal{R} + \mathcal{Q})x_0 = y.$$

Consequently,

$$\mathcal{J}^{-1}(\mathcal{R} + \mathcal{Q})\mathcal{J}\mathcal{J}^{-1}x_0 = \mathcal{J}^{-1}y,$$

and relation (3) implies that

$$BC\mathcal{J}^{-1}x_0 = A_2^{-1}A_1^{-1}\mathcal{J}^{-1}y,$$

or

$$(\mathcal{P}V(a, b)\mathcal{P} + \mathcal{Q})C\mathcal{J}^{-1}x_0 = A_2^{-1}A_1^{-1}\mathcal{J}^{-1}y.$$

Therefore, the element $X_0 = \mathcal{P}C\mathcal{J}^{-1}x_0$ is a solution of the equation (6) and since

$$\mathcal{J}^{-1} = \begin{pmatrix} I & I \\ J & -J \end{pmatrix},$$

the relation (5) follows. ■

Thus the solutions of the equation (4) generate solutions of equation (6) with the corresponding right-hand side Y . On the other hand, a similar reverse statement is also true.

Lemma 2.2 *If the equation (6) is solvable and if $X_0 = (\Phi, \Psi)$ is a solution of (6) with the right-hand side $Y = (g, h) \in H^p \times H^p$, then the element*

$$x_0 = \frac{1}{2}(\Phi - JQc\Phi + JQ\tilde{a}^{-1}\Psi, \Phi + JQc\Phi - JQ\tilde{a}^{-1}\Psi)^T, \quad (7)$$

where $c = \tilde{b}\tilde{a}^{-1}$, is a solution of the equation (4) with the right-hand side

$$y = \mathcal{P}\mathcal{J}A_1A_2Y. \quad (8)$$

Proof. If X_0 is a solution of equation (6), then

$$(\mathcal{P}V(a, b)\mathcal{P} + \mathcal{Q})X_0 = Y.$$

Applying the operator $\mathcal{J}A_1A_2$ to this equation, one obtains

$$\mathcal{J}A_1A_2(\mathcal{P}V(a, b)\mathcal{P} + \mathcal{Q})X_0 = \mathcal{J}A_1A_2Y.$$

The last expression can be rewritten as

$$(\mathcal{J}A_1A_2(\mathcal{P}V(a, b)\mathcal{P} + \mathcal{Q})C\mathcal{J}^{-1})\mathcal{J}C^{-1}X_0 = \mathcal{J}A_1A_2Y,$$

and the relation (3) implies that

$$R\mathcal{J}C^{-1}X_0 = \mathcal{J}A_1A_2Y.$$

Therefore, the element z_0 ,

$$z_0 = \mathcal{J}C^{-1}X_0 = \frac{1}{2}(\Phi - Jc\Phi + J\tilde{a}^{-1}\Psi, \Phi + Jc\Phi - J\tilde{a}^{-1}\Psi)^T$$

satisfies the equation

$$Rx = \mathcal{J}A_1A_2Y,$$

and since

$$Rz_0 = (\mathcal{R} + \mathcal{Q})z_0 = \mathcal{J}A_1A_2Y = \mathcal{P}\mathcal{J}A_1A_2Y + \mathcal{Q}\mathcal{J}A_1A_2Y,$$

the element $x_0 = \mathcal{P}z_0$ is a solution of equation (4). ■

As we will see later on, in some cases the equation (6) can be resolved in a closed form. Thus using the representation (7) one can also derive solutions φ of (2) from the solutions (Φ, Ψ) of the equation (6), provided that the right-hand side $Y \in H^p \times H^p$ in (6) is chosen in such a way that the first coordinate of the vector y in (8) is equal to the right hand side f of the equation (2). In this case, a solution of the equation (2) is given by the formula

$$\varphi_0 = \frac{1}{2}(\Phi - JQc\Phi + JQ\tilde{a}^{-1}\Psi). \quad (9)$$

It is clear that the choice of possible right-hand side Y in (6) is not unique. Nevertheless, it seems that the most suitable initial vector Y in the equation (6) is the one transferred into the vector $(f, 0)^T$ by the operator $\mathcal{P}\mathcal{J}A_1A_2$. Indeed, with such a choice of Y , the second equation in (4) would have the form

$$(T(a) - H(b))\psi = 0.$$

This equation is always solvable, so there will be no additional condition related to the solvability of the equation (4). In order to find such a right-hand side Y , one has to resolve the operator equation

$$\mathcal{P}\mathcal{J}A_1A_2Y = (f, 0)^T. \quad (10)$$

Setting $Y = (g, h)^T$, $g, h \in H^p$ one can write equation (10) as the system of equations

$$\begin{cases} g - PbPh - PaQJh = 2f \\ g - PbPh + PaQJh = 0, \end{cases} \quad (11)$$

with respect to unknown functions g and h . However, the problem of the determination of the functions g and h from the system (11) is equivalent to the solution of the equation

$$PaQJh = -2f,$$

which is not a simple task. Nevertheless, system (11) suggests a simple choice of the functions g and h which would lead to the equation (2) with the required right-hand side f , viz. one can consider the pair $g = 2f$, $h = 0$ so the corresponding right-hand side $Y = (2f, 0)$. Of course, a consequence of such a choice of the right hand side Y is that the equation

$$(T(a) - H(b))\psi = f$$

must be also solvable. Note that the solvability of the last equation is not directly connected to the solvability of (2), which means that if the operator $\mathcal{P}V(a, b)\mathcal{P}$ is not right invertible, our method will not work for some right hand sides f from the image of the operator $T(a) + H(b)$. However, the set of acceptable right-hand sides f is quite large because it is generated by those pairs $(f, 0)^T$, $f \in H^p$ which belongs to the image of the operator $\mathcal{P}V(a, b)\mathcal{P}$. More precisely, the following proposition holds.

Proposition 2.1 *If $Y := (2f, 0)^T$, $f \in H^p$ and the equation (6) with the right-hand side Y is solvable with the solution (Φ, Ψ) , then the equation (2) is also solvable and one of its solution can be written in the form (9).*

Proof. Straightforward computation. ■

Note that the connection between the solvability of the systems (4) and (6) with the corresponding right-hand sides $(f, f)^T$ and $(2f, 0)^T$ will be discussed later (see Remark 3.1 below).

3 Solution of non-homogeneous equations with Toeplitz plus Hankel operators.

In this section we construct solutions of the non-homogeneous equation (6) in the case where the generating functions $a, b \in L^\infty$ are connected in a special way. Thus let us assume that a and b satisfy the relation

$$a(t)\tilde{a}(t) = b(t)\tilde{b}(t), \quad t \in \mathbb{T}, \quad (12)$$

where $\tilde{a}(t) = a(1/t)$ and $\tilde{b}(t) = b(1/t)$, as before. In what follows this relation is called the matching condition and any duo (a, b) with the property (12) is called the matching pair. For any matching pair (a, b) one can construct another pair (c, d) of functions c and d defined by

$$c(t) := ab^{-1}(= \tilde{b}\tilde{a}^{-1}), \quad d := b\tilde{a}^{-1}(= \tilde{b}^{-1}a).$$

Such a pair (c, d) is called the subordinated pair for (a, b) , and the functions which constitutes a subordinated pair possess the property

$$c\tilde{c} = 1 = d\tilde{d}.$$

Let us also point out that if (c, d) is the subordinated pair for a matching pair (a, b) , then (\bar{d}, \bar{c}) is the subordinated pair for the matching pair (\bar{a}, \bar{b}) which define the adjoint operator

$$(T(a) + H(b))^* := T(\bar{a}) + H(\bar{b}),$$

for the Toeplitz plus Hankel operator $T(a) + H(b)$.

Consider now the equation

$$T(V(a, b))X = (2f, 0)^T, \quad f \in H^p, \quad (13)$$

where $T(V(a, b)) = \mathcal{P}V(a, b)\mathcal{P}$ is the Toeplitz operator generated by the matrix $V(a, b)$. If (a, b) is a matching pair, then $V(a, b)$ is a triangular matrix, namely,

$$V(a, b) = \begin{pmatrix} 0 & d \\ -c & \tilde{a}^{-1} \end{pmatrix},$$

where (c, d) is the corresponding subordinated pair.

It turns out that in this case, the solutions of the matrix equation (13) depend on the solutions of equations with Toeplitz operators $T(c)$ and $T(d)$. Therefore, we would like to recall some facts concerning scalar Toeplitz operators.

Definition 3.1 *A function $a \in L^\infty$ admits a weak Wiener–Hopf factorization in H^p , if it can be represented in the form*

$$a = a_- t^n a_+, \quad (14)$$

where $n \in \mathbb{Z}$, $a_+ \in H^q$, $a_+^{-1} \in H^p$, $a_- \in \overline{H^p}$, $a_-^{-1} \in \overline{H^q}$, and $a_-(\infty) = 1$.

The weak Wiener–Hopf factorization of a function a is unique, if it exists. The functions a_- and a_+ are called the factorization factors, and the number n is the factorization index. If $a \in L^\infty$ and the operator $T(a)$ is Fredholm, then the function a admits the weak Wiener–Hopf factorization with $n = -\text{ind } T(a)$ [2, 9]. Moreover, in this case, the factorization factors possess an additional property—viz. the linear operator $a_+^{-1} P a_-^{-1} I$ defined on $\text{span } \{t^k : k \in \mathbb{Z}_+\}$ can be boundedly extended on the whole space H^p . Throughout this paper, such a kind of the weak Wiener–Hopf factorization in H^p is called simply Wiener–Hopf factorization in H^p . The following result is well-known.

Theorem 3.1 (see [2]) *If $a \in L^\infty$, then Toeplitz operator $T(a) : H^p \rightarrow H^p$, $1 < p < \infty$ is Fredholm and $\text{ind } T(a) = -n$ if and only if the generating function a admits the Wiener–Hopf factorization (14) in H^p .*

Thus if $T(a)$ is Fredholm and $n \geq 0$, then the operator $T(a)$ is left invertible and

$$T_l^{-1}(a) = T(t^{-n})T(a_+^{-1})T(a_-^{-1})$$

is a left inverse of $T(a)$, whereas for $n \leq 0$, the operator $T(a)$ is right invertible and

$$T_r^{-1}(a) = T(a_+^{-1})T(a_-^{-1})T(t^{-n}) \quad (15)$$

is one of its right inverses. For definiteness, in the following the notation $T_r^{-1}(a)$ is always used for the operator defined by (15).

In this work we mainly use Toeplitz operators generated by matching functions, i.e. by the functions satisfying the condition $g\tilde{g} = 1$. For such functions, factorization (14) has special properties. Thus if $T(g) : H^p \rightarrow H^p$, $1 < p < \infty$ is a Fredholm operator, then it was shown in [6] that the matching function g can be represented as follows

$$g(t) = g_+(t)t^n (\sigma(g)\tilde{g}_+^{-1}(t)), \quad (16)$$

where $\sigma(g) := g_+(0) = \pm 1$ is called the factorization signature. In [6], there is discussed how to find the factorization signature in some particular cases. For instance, if g is continuous at the point 1 then $\sigma(g) = g(1)$.

Assume now that the operators $T(c)$ and $T(d)$ are Fredholm and let $\kappa_1 = \kappa_c := \text{ind } T(c)$ and $\kappa_2 = \kappa_d := \text{ind } T(d)$. If $T(c)$ is a right invertible operator, then we also consider the operator W defined by

$$W\varphi := T_r^{-1}(c)T(\tilde{a}^{-1})\varphi - JQcPT_r^{-1}(c)T(\tilde{a}^{-1})\varphi + JQ\tilde{a}^{-1}\varphi.$$

Now we can describe the solution set for the equation (2). The structure of this set depends on the indices of the operators $T(c)$ and $T(d)$. Let us start with the case where κ_c and κ_d are non-negative.

Theorem 3.2 *If $\kappa_c \geq 0$ and $\kappa_d \geq 0$, then for any $f \in H^p$ the equation (2) is solvable and all solutions of this equation are given by the formula*

$$\begin{aligned} \varphi = & T_r^{-1}(c)T(\tilde{a}^{-1})T_r^{-1}(d)f - JQcT_r^{-1}(c)T(\tilde{a}^{-1})T_r^{-1}(d)f + JQ\tilde{a}^{-1}T_r^{-1}(d)f \\ & + \text{sgn}(\kappa_c) c_+^{-1} \sum_{k=0}^{\kappa(1)} r_k^{(1)} u_k^{(\kappa(1), -)} + \text{sgn}(\kappa_d) W \left(d_+^{-1} \sum_{k=0}^{\kappa(2)} r_k^{(2)} u_k^{(\kappa(2), +)} \right), \end{aligned} \quad (17)$$

where $r_k^{(i)}$, $k = 0, 1, \dots, k(i)$, $i = 1, 2$, are arbitrary complex numbers,

$$\begin{aligned} \kappa(i) = m_i - 1, \quad u_k^{(\kappa(i), \pm)}(t) &= t^{m_i - k - 1} \pm \sigma(s_i) t^{m_i + k}, \quad \text{if } \kappa_i = 2m_i, \quad i = 1, 2, \\ \kappa(i) = m_i, \quad u_k^{(\kappa(i), \pm)}(t) &= t^{m_i + k} \pm \sigma(s_i) t^{m_i - k}, \quad \text{if } \kappa_i = 2m_i + 1, \quad i = 1, 2, \end{aligned}$$

$$\text{sgn}(r) := \begin{cases} 0 & \text{if } r = 0, \\ 1 & \text{if } r > 0, \end{cases}$$

and $s_1 = c$, $s_2 = d$.

Proof. If $\kappa_c \geq 0$ and $\kappa_d \geq 0$, the operators $T(c)$ and $T(d)$ are right invertible. Moreover, the corresponding operator $T(V(a, b))$ is also right-invertible and one can easily check that the operator $U : H^p \times H^p \rightarrow H^p \times H^p$,

$$U = \begin{pmatrix} T_r^{-1}(c)T(\tilde{a}^{-1})T_r^{-1}(d) & -T_r^{-1}(c) \\ T_r^{-1}(d) & 0 \end{pmatrix} \quad (18)$$

is one of the right-inverses for $T(V(a, b))$. Hence, one of the solutions of equation (13) is

$$(\Phi, \Psi)^T = U((2f, 0)^T) = (2T_r^{-1}(c)T(\tilde{a}^{-1})T_r^{-1}(d)f, 2T_r^{-1}(d)f)^T,$$

and using formula (9), one obtains a solution of the non-homogeneous equation (2). Thus, if $\kappa_c = \kappa_d = 0$, the unique solution of the equation (2) is

$$\begin{aligned} \varphi_0 &= \frac{1}{2}(\Phi - JQc\Phi + JQ\tilde{a}^{-1}\Psi) \\ &= T_r^{-1}(c)T(\tilde{a}^{-1})T_r^{-1}(d)f - JQcT_r^{-1}(c)T(\tilde{a}^{-1})T_r^{-1}(d)f + JQ\tilde{a}^{-1}T_r^{-1}(d)f. \end{aligned}$$

On the other hand, in the case where at least one of the indices κ_c or κ_d does not vanish, the function φ_0 is a partial solution of (2). In order to obtain all solutions, one has to add the solutions of the corresponding homogeneous equation. However, the kernels of the operators $T(a) + H(b)$ have been described earlier. Thus using Proposition 3.7 and Theorem 5.4 of [6], one arrives at the representation (17). ■

Now let us consider the situation where κ_c and κ_d are non-positive.

Theorem 3.3 *Let $\kappa_c \leq 0$ and $\kappa_d \leq 0$, and let c_- , d_- be factorization factors in the Wiener–Hopf factorization of the matching functions c and d , respectively. If function $f \in H^p$ satisfies the conditions*

$$\begin{aligned} \int_{\mathbb{T}} \overline{d_-^{-1}(t)} t^j \overline{f(t)} |dt| &= 0, \quad j = 0, 1, \dots, -\kappa_d - 1, \\ \int_{\mathbb{T}} T_r^{-1}(\bar{d})T(\tilde{a}^{-1}) \left(\overline{c_-^{-1}(t)} t^j \right) \overline{f(t)} |dt| &= 0, \quad j = 0, 1, \dots, -\kappa_c - 1, \end{aligned} \quad (19)$$

then the equation (2) is uniquely solvable and its solution φ_0 has the form

$$\varphi_0 = T_l^{-1}(c)T(\tilde{a}^{-1})T_l^{-1}(d)f - JQcT_l^{-1}(c)T(\tilde{a}^{-1})T_l^{-1}(d)f + JQ\tilde{a}^{-1}T_l^{-1}(d)f. \quad (20)$$

Proof. If indices κ_c and κ_d are non-positive, the operator $T(V(a, b))$ is left-invertible and

$$T_l^{-1}(V(a, b)) = \begin{pmatrix} T_l^{-1}(c)T(\tilde{a}^{-1})T_l^{-1}(d) & -T_l^{-1}(c) \\ T_l^{-1}(d) & 0 \end{pmatrix}$$

is one of its left inverses. Therefore, equation (13) is solvable if and only if its right-hand side $(2f, 0)^T$ is orthogonal to any solution of the equation

$$T^*(V(a, b))\psi = 0, \quad (21)$$

where

$$T^*(V(a, b)) = \begin{pmatrix} 0 & -T(\bar{c}) \\ T(\bar{d}) & T(\tilde{a}^{-1}) \end{pmatrix}$$

is the adjoint operator for the operator $T(V(a, b))$. However, according to [6, Proposition 3.3], the kernel of this operator can be represented in the form

$$\ker T^*(V(a, b)) = \Omega(\bar{d}) + \widehat{\Omega}(\bar{c}),$$

where

$$\begin{aligned}\Omega(\bar{d}) &:= \{(v, 0)^T : v \in \ker T(\bar{d})\}, \\ \widehat{\Omega}(\bar{c}) &:= \{(T_r^{-1}(\bar{d})T(\bar{a}^{-1})u, u)^T : u \in \ker T(\bar{c})\}.\end{aligned}$$

Taking into account the fact that

$$\begin{aligned}\ker T(\bar{d}) &= \{\bar{d}_-^{-1} t^j : j = 0, 1, \dots, -\kappa_d - 1\}, \\ \ker T(\bar{c}) &= \{\bar{c}_-^{-1} t^j : j = 0, 1, \dots, -\kappa_c - 1\},\end{aligned}$$

one obtains solvability conditions (12). If equation (13) is solvable, its solution ψ is

$$\psi = T_L^{-1}(V(a, b))(2f, 0)^T = (2T_l^{-1}(c)T(\bar{a}^{-1})T_l^{-1}(d)f, 2T_l^{-1}(d)f)^T,$$

and formula (9) leads to the representation (20). ■

Remark 3.1 *Using Theorem 5.4 and Theorem 6.4 of [6], one can obtain conditions of simultaneous solvability of the equations*

$$\begin{aligned}(T(a) + H(b))\varphi &= f, \\ (T(a) - H(b))\psi &= f,\end{aligned}$$

and show that if these two equations are solvable, then the conditions (19) are satisfied, so the system (13) is also solvable.

Consider one more case where all solution of the equation (2) can be found.

Theorem 3.4 *Let $\kappa_c > 0$, $\kappa_d < 0$, and let c_- , d_- be the factorization factors in the Wiener–Hopf factorization of the matching functions c and d , respectively. If function $f \in H^p$ satisfies the conditions*

$$\int_{\mathbb{T}} \overline{d_-^{-1}(t)} t^j \overline{f(t)} |dt| = 0, \quad j = 0, 1, \dots, -\kappa_d - 1, \quad (22)$$

then the equation (2) is solvable and all solutions of this equation are given by the formula

$$\begin{aligned}\varphi &= T_r^{-1}(c)T(\bar{a}^{-1})T_l^{-1}(d)f - JQcT_r^{-1}(c)T(\bar{a}^{-1})T_l^{-1}(d)f \\ &\quad + JQ\bar{a}^{-1}T_l^{-1}(d)f + c_+^{-1} \sum_{k=0}^{\kappa(1)} r_k^{(1)} u_k^{(\kappa(1), -)},\end{aligned} \quad (23)$$

where $\kappa(1), r_k^{(1)}$, and $u_k^{(\kappa(1), -)}$ are defined in Theorem 3.2.

Proof. Taking into account the conditions imposed on the indices κ_c and κ_d , one obtains that the operators $T(c)$ and $T(d)$ are respectively, right- and left-invertible. Further, straightforward computations show that the operator

$$T_g^{-1}(V(a, b)) = \begin{pmatrix} T_r^{-1}(c)T(\tilde{a}^{-1})T_l^{-1}(d) & -T_r^{-1}(c) \\ T_l^{-1}(d) & 0 \end{pmatrix},$$

is a generalized inverse for the operator $T(V(a, b))$. Hence, equation (18) is solvable if and only if its right hand side $(2f, 0)^T$ is orthogonal to all solutions of the equation (21). On the other hand, the operator $T^*(V(a, b))$ admits the factorizations.

$$\begin{aligned} T^*(V(a, b)) &= \begin{pmatrix} 0 & -T(\bar{c}) \\ T(\bar{d}) & T(\tilde{a}^{-1}) \end{pmatrix} \\ &= \begin{pmatrix} T(\bar{c}) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & T(\tilde{a}^{-1}) \end{pmatrix} \begin{pmatrix} -T(\bar{d}) & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \quad (24)$$

Note that the first operator in (24) is invertible from the left, whereas the operator

$$D := \begin{pmatrix} 0 & -I \\ I & T(\tilde{a}^{-1}) \end{pmatrix}$$

is just invertible and

$$D^{-1} = \begin{pmatrix} T(\tilde{a}^{-1}) & I \\ -I & 0 \end{pmatrix}.$$

Therefore, factorization (24) shows that the kernel of the operator $T^*(V(a, b))$ can be represented in the form

$$\ker T^*(V(a, b)) = \{(v, 0)^T; v \in \ker T(\bar{d})\},$$

and the solvability condition (22) follows. A partial solution of the equation (13) can be now obtained, viz.

$$(\Phi, \Psi)^T = T_g^{-1}(V(a, b))(2f, 0)^T,$$

which allows one to construct a partial solution of the equation (2). Adding the solutions of the corresponding homogeneous equation, one obtains formula (23). ■

Consider now the last case remaining—viz. the situation where $\kappa_c < 0$ and $\kappa_d > 0$. The method we use is not directly applicable here. However, the initial equation admits a modification which can be employed to find an analytic solution of the equation (2). Before we go on, let us formulate a few auxiliary results.

Lemma 3.1 *Let $f \in H^p$ and $n \in \mathbb{N}$. If the equation*

$$(T(a) + H(b))\varphi = f \quad (25)$$

is solvable and φ_0 is a solution of (25), then the equation

$$(T(t^{-n}a) + H(t^n b))\psi = f \quad (26)$$

is also solvable and there is a solution ψ_0 of (26) which belongs to the image $\text{im } T(t^n)$ of the operator $T(t^n) : H^p \rightarrow H^p$.

The proof is straightforward and is based on the relation

$$(T(a) + H(b)) = (T(t^{-n}a) + H(t^n b))T(t^n). \quad (27)$$

Note that one of the solutions in question is $\psi_0 = T(t^n)\varphi_0$. It is also worth mentioning that

$$\text{im } T(t^n) = \{\psi \in H^p : \widehat{\psi}_0 = \widehat{\psi}_1 = \dots = \widehat{\psi}_{n-1} = 0\},$$

where $\widehat{\psi}_j$, $j = 0, 1, \dots, n-1$ are the Fourier coefficients of the function ψ .

Corollary 3.1 *Let $f \in H^p$ and $n \in \mathbb{N}$. If equation (26) does not have any solutions $\psi \in \text{im } T(t^n)$, then equation (25) is not solvable.*

Thus any solution of (25) produces a solution of (26) which lies in the set $\text{im } T(t^n)$. On the other hand, one also has a reverse statement.

Lemma 3.2 *Let $f \in H^p$ and $n \in \mathbb{N}$. If the equation (26) is solvable and has a solution $\psi_0 \in \text{im } T(t^n)$, then equation (25) is also solvable and one of its solutions has the form*

$$\varphi_0 = T(t^{-n})\psi_0. \quad (28)$$

Proof. Use the fact that $T(t^n)T(t^{-n}) : \text{im } T(t^n) \rightarrow \text{im } T(t^n)$ is the identity operator.

■

Now we can describe how to find the solutions of equation (25) in the case at hand. Assume that this equation is solvable and choose an $n \in \mathbb{N}$ such that

$$1 \geq 2n + \kappa_c \geq 0.$$

Such an n is uniquely defined and

$$2n + \kappa_c = \begin{cases} 0, & \text{if } \kappa_c \text{ is even,} \\ 1, & \text{if } \kappa_c \text{ is odd.} \end{cases}$$

According to Lemma 3.1 equation (26) is also solvable. One can also see that (at^{-n}, bt^n) is again a matching pair with the subordinated pair (ct^{-2n}, d) . Moreover,

the indices of the corresponding operators $\tilde{\kappa}_1 = \text{ind } T(ct^{-2n}) = \kappa_c + 2n$ and $\tilde{\kappa}_2 = \text{ind } T(d) = \kappa_d$ are non-negative. Thus the operator $T(at^{-n}) + H(bt^n)$ satisfies all conditions of Theorem 3.2, and using the correspondingly adapted formula (17), one obtains all solutions of the equation (26). Since equation (25) is solvable, the set of solutions of (26) contains at least one solution ψ_0 which belongs to $\text{im } T(t^n)$. Now one can employ formula (28) to obtain a solution of (25). The set of the solutions of the corresponding homogeneous equation

$$(T(a) + H(b))\varphi = 0$$

is described in Theorem 6.3 of [6]. This description allows one to find all solutions of the equation (25) in the case $\kappa_c < 0$, $\kappa_d > 0$.

4 Examples

Let us illustrate the above theory by a few simple examples.

Example 4.1 *Consider the equation*

$$(T(t^{-2}) + H(t^2))\varphi(t) = f(t). \quad (29)$$

Obviously, the functions $a(t) = t^{-2}$ and $b(t) = t^2$ constitute a matching pair. Moreover, one has

$$\begin{aligned} \tilde{a}^{-1}(t) &= t^{-2}, & c(t) &= a(t)b^{-1}(t) = t^{-4}, & d(t) &= \tilde{a}^{-1}(t)b(t) = 1, \\ c_+(t) &= c_-(t) = 1, & d_+(t) &= d_-(t) = 1, & \sigma(c) &= \sigma(d) = 1. \\ \kappa_c &= 4, & \kappa_d &= 0. \end{aligned}$$

Thus equation (29) is subject to Theorem 3.2. It is solvable for any right-hand side $f \in H^p$, $1 < p < \infty$ and any solution of (29) can be obtained from formula (17). Taking into account that $T_r^{-1}(c) = Pt^4P$, $T^{-1}(d) = I$, the solutions of this equation can be written in the form

$$\varphi(t) = Pt^4Pt^{-2}Pf(t) - JQt^{-4}Pt^4Pt^{-2}Pf(t) + JQt^{-2}f(t) + r_1(t - t^2) + r_2(t^2 - t^3), \quad (30)$$

where r_1, r_2 are arbitrary complex numbers. In particular, let us find a solution of the equation (29) for a given right-hand side f . For example, if $f(t) = t^6 + 3t^4$ and $r_1 = 0$, $r_2 = 0$, the formula (30) produces the function

$$\varphi(t) = t^8 + 3t^6,$$

and one can easily check that this is a partial solution of the equation (29) with the right-hand side $f(t) = t^6 + 3t^4$.

Example 4.2 Consider the equation

$$(T(2t + 1) + H(2t + 1))\varphi(t) = f(t). \quad (31)$$

It is clear that $(2t + 1, 2t + 1)$ is a matching pair. Further, one has

$$\tilde{a}^{-1}(t) = \frac{t}{t+2}, \quad c(t) = 1, \quad d(t) = \frac{t(2t+1)}{t+2},$$

and the function $d(t)$ admits the factorization

$$d(t) = \left(\frac{2}{t+2} \right) t^2 \left(\frac{2t+1}{2t} \right),$$

with the factorization factors

$$d_+(t) = \frac{2}{t+2}, \quad d_-(t) = \frac{2t+1}{2t}.$$

Thus

$$\kappa_c = 0, \quad \kappa_d = -2,$$

and the equation (31) is subject to Theorem 3.3. Therefore, on order to use the above method, we need the right-hand side f to satisfy the condition

$$\int_{\mathbb{T}} \frac{t^j}{t+2} \overline{f(t)} |dt| = 0, \quad j = 0, 1. \quad (32)$$

If these conditions are satisfied, then equation (31) is uniquely solvable and its solution can be found by the formula (20). Consider, for example, the function $f(t) = (2t + 1)(t^2 + t)$. One can easily check that this function f satisfies the solvability conditions (32). Applying formula (20) with

$$T_l^{-1}(d) = Pt^{-2}P(t+2)P\left(\frac{t}{2t+1}\right)P,$$

one obtains the function

$$\varphi_0(t) = t(t+1),$$

which is the solution of (31) with the right-hand side $f(t) = (2t + 1)(t^2 + t)$.

On the other hand, it is worth mentioning that in this case the operator $T(V(a, b))$ is not right invertible. Therefore, there are right-hand sides $f \in H^p$ such that the equation (31) is solvable but its solution cannot be found by the method used.

Example 4.3 Let $b = b(t)$ be a matching function, that is $b(t)\widetilde{b}(t) = 1$. Consider the following equation

$$\varphi + H(b)\varphi = f, \quad f \in H^p. \quad (33)$$

Assume that the operator $T(b)$ is Fredholm. Then according to (16) the function b admit the Wiener–Hopf factorization of the form

$$b(t) = b_+(t)t^n(\boldsymbol{\sigma}(b)\widetilde{b}_+^{-1}(t)). \quad (34)$$

One also has $a(t) = 1, t \in \mathbb{T}$, so that

$$c(t) = b^{-1}(t) = \widetilde{b}(t), \quad d(t) = b(t).$$

Therefore, d has the factorization (34) and c can be factorized as follows

$$c(t) = b_+^{-1}(t)t^{-n}(\boldsymbol{\sigma}(b)\widetilde{b}_+(t)).$$

Thus

$$\kappa_c = n, \quad \kappa_d = -n,$$

and if $n \neq 0$, the indices κ_c and κ_d have different signs. Assume for definiteness that $n \geq 0$. Then we are in the situation described by Theorem 3.4, and if

$$\int_{\mathbb{T}} \widetilde{b}_+(t) \overline{f(t)} t^j |dt| = 0, \quad j = 0, 1, \dots, n-1,$$

one can write the solutions in the form (23). In particular, one has

$$\begin{aligned} T_r^{-1}(c) &= \boldsymbol{\sigma}(b)T(b_+)T(\widetilde{b}_+^{-1})T(t^n) \\ T_l^{-1}(d) &= \boldsymbol{\sigma}(b)T(t^{-n})T(b_+^{-1})T(\widetilde{b}_+). \end{aligned}$$

Moreover, let Q_n be the operator defined by

$$Q_n\varphi(t) = Q_n \left(\sum_{j=0}^{\infty} \widehat{\varphi}_j t^j \right) := \sum_{j=n+1}^{\infty} \widehat{\varphi}_j t^j.$$

One can easily see that $Q_n = T(t^n)T(t^{-n})$. Therefore, all solutions of the equation (33) can be written in the form

$$\varphi = (I - JQb^{-1})T(b_+)T(\widetilde{b}_+^{-1})Q_nT(b_+^{-1})T(\widetilde{b}_+)f + b_+ \sum_{j=0}^{\kappa(1)} r_j u_j^{(k(1), -)},$$

where $r_j, j = 0, 1, \dots, \kappa(1)$ are arbitrary complex numbers.

References

- [1] BASOR, E. L., AND EHRHARDT, T. Fredholm and invertibility theory for a special class of Toeplitz + Hankel operators. *J. Spectral Theory* 3, 3 (2013), 171–214.
- [2] BÖTTCHER, A., AND SILBERMANN, B. *Analysis of Toeplitz operators*, second ed. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006. Prepared jointly with Alexei Karlovich.
- [3] DIDENKO, V. D., AND SILBERMANN, B. Index calculation for Toeplitz plus Hankel operators with piecewise quasi-continuous generating functions. *Bull. London Math. Soc.* 45, 3 (2013), 633–650.
- [4] DIDENKO, V. D., AND SILBERMANN, B. The Coburn-Simonenko Theorem for some classes of Wiener–Hopf plus Hankel operators. *Publications de l’Institut Mathe’matique* 96 (110) (2014), 85–102.
- [5] DIDENKO, V. D., AND SILBERMANN, B. Some results on the invertibility of Toeplitz plus Hankel operators. *Ann. Acad. Sci. Fenn. Math.* 39, 1 (2014), 443–461.
- [6] DIDENKO, V. D., AND SILBERMANN, B. Structure of kernels and cokernels of Toeplitz plus Hankel operators. *Integral Equations Operator Theory* 80, 1 (2014), 1–31.
- [7] DUDUCHAVA, R. V. *Integral equations in convolution with discontinuous presymbols, singular integral equations with fixed singularities, and their applications to some problems of mechanics*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1979.
- [8] KARAPETIANTS, N., AND SAMKO, S. *Equations with involutive operators*. Birkhäuser Boston Inc., Boston, MA, 2001.
- [9] LITVINCHUK, G. S., AND SPITKOVSKII, I. M. *Factorization of measurable matrix functions*, vol. 25 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1987.
- [10] MEISTER, E., SPECK, F.-O., AND TEIXEIRA, F. S. Wiener-Hopf-Hankel operators for some wedge diffraction problems with mixed boundary conditions. *J. Integral Equations Appl.* 4, 2 (1992), 229–255.
- [11] ROCH, S., SANTOS, P. A., AND SILBERMANN, B. *Non-commutative Gelfand theories*. Universitext. Springer-Verlag London Ltd., London, 2011. A tool-kit for operator theorists and numerical analysts.

- [12] ROCH, S., AND SILBERMANN, B. *Algebras of convolution operators and their image in the Calkin algebra*, vol. 90 of *Report MATH*. Akademie der Wissenschaften der DDR, Karl-Weierstrass-Institut für Mathematik, Berlin, 1990.
- [13] ROCH, S., AND SILBERMANN, B. A handy formula for the Fredholm index of Toeplitz plus Hankel operators. *Indag. Math.* 23, 4 (2012), 663–689.
- [14] SILBERMANN, B. The C^* -algebra generated by Toeplitz and Hankel operators with piecewise quasicontinuous symbols. *Integral Equations Operator Theory* 10, 5 (1987), 730–738.